

FINITE DIMENSIONAL SUBSPACES OF NONCOMMUTATIVE L_p SPACES.

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ABSTRACT. We prove the following noncommutative version of Lewis's classical result. Every n -dimensional subspace E of $L_p(\mathcal{M})$ ($1 < p < \infty$) for a von Neumann algebra \mathcal{M} satisfies

$$d_{cb}(E, RC_{p'}^n) \leq c_p \cdot n^{\left|\frac{1}{2} - \frac{1}{p}\right|}$$

for some constant c_p depending only on p , where $\frac{1}{p} + \frac{1}{p'} = 1$ and $RC_{p'}^n = [R_n \cap C_n, R_n + C_n]_{\frac{1}{p'}}$. Moreover, there is a projection $P : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$ onto E

with $\|P\|_{cb} \leq c_p \cdot n^{\left|\frac{1}{2} - \frac{1}{p}\right|}$. We follow the classical change of density argument with appropriate noncommutative variations in addition to the opposite trick.

1. INTRODUCTION

Lewis showed that ([13]) for any finite dimensional subspace of a commutative L_p space we can find a special basis forming an orthonormal system with respect to a certain change of density, namely a power of their square function. More precisely, for a n -dimensional subspace X of $L_p(\mu)$ ($1 \leq p < \infty$) for some measure μ there is a basis $(x_i)_{i=1}^n$ of X satisfying

$$(1.1) \quad \int X^{p-2}(t) x_i^*(t) x_j(t) d\mu(t) = \delta_{ij} \text{ for any } 1 \leq i, j \leq n,$$

where $X(t) = \left(\sum_{i=1}^n x_i^2(t)\right)^{\frac{1}{2}}$ and $\int X^p(t) d\mu(t) = n$.

This change of density phenomenon was used to prove the following estimate of the Banach-Mazur distance from the Hilbert space with the same dimension and the relative projection constant with respect to L_p . Indeed, for $1 < p < \infty$ we have

$$(1.2) \quad d(X, \ell_2^n) \leq n^{\left|\frac{1}{2} - \frac{1}{p}\right|},$$

where $d(Y, Z) = \inf\{\|T\| \|T^{-1}\| \mid T : Y \rightarrow Z, \text{ isomorphism}\}$ and

$$\lambda(X, L_p(\mu)) \leq n^{\left|\frac{1}{2} - \frac{1}{p}\right|},$$

where $\lambda(Y, Z) := \inf\{\|P\| \mid P : Z \rightarrow Y, \text{ projection onto } Y\}$ for $Y \subseteq Z$.

Now, it is natural to be interested in what can be said for finite dimensional subspaces of noncommutative L_p (in the sense of Haagerup). However, we are not interested in Banach space structure since we already know that the same results as (1.2) hold by θ -Hilbertian space approach of Pisier ([19, 20]). Recall that a Banach space X is called θ -Hilbertian ($0 < \theta < 1$) if X is a complex interpolation between a Banach space and a Hilbert space with the parameter of θ , and it is known that any n -dimensional subspace Z of a θ -Hilbertian satisfies $d(Z, \ell_2^n) \leq n^{\frac{1}{2}(1-\theta)}$. Note that $L_p = [L_1, L_2]_{\frac{2}{p}}$ for $1 < p < 2$ and $L_p = [L_\infty, L_2]_{\frac{2}{p}}$ for $2 \leq p < \infty$.

Since there is a canonical operator space structure on L_p ([17, 18]) it is again natural to move our attention to their operator space structure. Let E be a n -dimensional subspace of $L_p(\mathcal{M})$, where \mathcal{M} is a von Neumann algebra. Following

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the approach of Lewis we will prove that there is a basis $(x_i)_{i=1}^n$ of E satisfying an orthonormal condition similar to (1.1) at least when \mathcal{M} is semifinite. In this case, we will use the change of density with respect to $(\sum_{i=1}^n x_i^* x_i)^{\frac{p-2}{2}}$, a power of the column square function of $(x_i)_{i=1}^n$.

Then, we would be able to show the following main results of this paper.

Theorem. *Let E be n -dimensional subspace of $L_p(\mathcal{M})$ for a von Neumann algebra \mathcal{M} . Then we have*

$$d_{cb}(E, RC_{p'}^n) \leq c_p \cdot n^{|\frac{1}{2} - \frac{1}{p}|}$$

for some constant c_p depending only on p .

Corollary. *Let E , \mathcal{M} , and c_p as before. Then we have $\lambda_{cb}(E, L_p(\mathcal{M})) \leq c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}$.*

In the above $RC_{p'}^n$ is the n -dimensional Hilbertian operator space defined by $RC_{p'}^n = [R_n \cap C_n, R_n + C_n]_{\frac{1}{p'}}$ ([18]), where R_n and C_n are the n -dimensional row and column Hilbert spaces, respectively. and $\lambda_{cb}(\cdot, \cdot)$ is the relative cb-projection constant defined by

$$\lambda_{cb}(F, G) := \inf\{\|P\|_{cb} \mid P : G \rightarrow F, \text{ projection onto } F\}$$

for operator spaces $F \subseteq G$. The result of the theorem is sharp in the sense that there is a n -dimensional operator space E with $d_{cb}(E, RC_{p'}^n) \geq C \cdot n^{\frac{1}{2} - \frac{1}{p}}$ for some constant C independent of n . Actually, it is easy to see that $E = R_p^n$ is an example. Note also that the case $p = 1$ and $p = \infty$ are covered by a more general result of Junge (Proposition 3.3.2.4 of [5] or Corollary 7.9 of [18]).

This paper is organized as follows. In section 2, we present technical tools for the proof of the main results, namely, the support of elements in $L_p(\mathcal{M})$ for semifinite von Neumann algebra \mathcal{M} , the equality condition in Hölder's inequality for semifinite von Neumann algebras, Haagerup's reduction principle, the operator space structure of $L_p(\mathcal{M})^{op}$, the opposite of $L_p(\mathcal{M})$ for σ -finite von Neumann algebra \mathcal{M} , and an automatic complete boundedness result. In section 3, we restrict ourselves to the case when the underlying von Neumann algebra is semifinite and prove that a special basis of a finite dimensional subspace of $L_p(\mathcal{M})$ can be chosen as in the commutative case. With the chosen basis and the change of density argument we get a factorization result, which will be extended to the general von Neumann algebra case. In the final section, we prove the main results using the previous factorization and the opposite argument.

Throughout this paper, we assume that the reader is familiar with basic concepts in operator spaces ([2, 18]), operator algebras ([21, 22, 9, 10]), and noncommutative L_p spaces ([22, 23, 8]).

For $1 \leq p \leq \infty$, S_p (resp. S_p^n) refers to the Schatten p -class on ℓ_2 (resp. ℓ_2^n), and for an operator space E , $S_p(E)$ (resp. $S_p^n(E)$) implies its vector valued version ([17]). Let R_p and C_p be the row and column of S_p defined by $\overline{\text{span}}\{e_{1i} : i \geq 1\}$ and $\overline{\text{span}}\{e_{i1} : i \geq 1\}$ in S_p , respectively, and R_p^n and C_p^n are their n -dimensional versions.

In some special cases vector valued L_p spaces can be defined for general von Neumann algebras. Let $L_p(\mathcal{M}; C_p)$ (resp. $L_p(\mathcal{M}; R_p)$) be the space of sequences $(x_i)_{i \geq 1}$ in $L_p(\mathcal{M})$ with the norm

$$\begin{aligned} \|(x_i)_{i \geq 1}\|_{L_p(\mathcal{M}; C_p)} &= \left\| \left(\sum_i x_i^* x_i \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M})} \\ (\text{resp. } \|(x_i)_{i \geq 1}\|_{L_p(\mathcal{M}; R_p)} &= \left\| \left(\sum_i x_i x_i^* \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M})} \end{aligned}$$

Note that $L_p(\mathcal{M}; C_p)^* = L_p(\mathcal{M}; C_{p'})$ for $1 \leq p < \infty$.

For operator space E and F , $E \oplus_p F$ implies the direct sum in the sense of ℓ_p ($1 \leq p \leq \infty$). Then $E \cap_p F$ and $E +_p F$, the operator space intersection and sum in the sense of ℓ_p , are defined by

$$E \cap_p F = \{(x, x) : x \in E \cap F\} \text{ and } E +_p F = E \oplus_p F / \Delta,$$

where $\Delta = \{(x, -x) : x \in E \cap F\}$. We simply write $E \cap F$ and $E + F$ instead of $E \cap_\infty F$ and $E +_1 F$, respectively. Note that $E \oplus_p F$'s are all completely isomorphic with the constant independent of $1 \leq p \leq \infty$. Furthermore, we have the following completely isomorphism with the constant independent of $1 \leq p \leq \infty$.

$$RC_p \cong \begin{cases} R_p + C_p & (1 \leq p < 2), \\ R_p \cap C_p & (2 \leq p \leq \infty) \end{cases},$$

where $RC_p = [R \cap C, R + C]_{\frac{1}{p}}$.

2. TECHNICAL TOOLS

In this section we present essential tools which we will need in the sequel.

We start with the support of elements in $L_p(\mathcal{M})$, where \mathcal{M} is a semifinite von Neumann algebra with a normal semifinite faithful (shortly n.s.f.) trace τ . See chapter 9 of [22] or chapter 1 of [23] for the definition of $L_p(\mathcal{M})$. Let $\mathcal{M} \subset B(\mathcal{H})$ for some Hilbert space \mathcal{H} and $a \in L_p(\mathcal{M})$. Then a can be understood as a closed densely defined operator on \mathcal{H} . Thus, we have a unique polar decomposition (Theorem 6.1.11 of [10])

$$a = v |a|,$$

where v is a partial isometry in \mathcal{M} whose initial space is the closure of the range of $|a|$. Then, the support projection q of a (we write $q = \text{supp}(a)$) is defined by $q = v^* v$. Using functional calculus we have (Theorem 5.6.26 of [9] and Proposition 8 of [23])

$$q = 1_{(0, \infty)}(|a|).$$

Now we consider a positive element $X \in L_p^+(\mathcal{M})$. Then, by functional calculus again X and $X^p \in L_1^+(\mathcal{M})$ has the same support $q = 1_{(0, \infty)}(X)$, and

$$(2.1) \quad qX^r = X^r q = X^r$$

for any $r > 0$. If we set $\phi(\cdot) = \tau(\cdot X_p)$, then ϕ is a positive normal functional on \mathcal{M} and faithful on $q\mathcal{M}q$ (p. 455 of [10]), since q is the least projection satisfying $qX^p = X^p q = X^p$. Thus, X^p , or equivalently X , is invertible as an operator affiliated to $q\mathcal{M}q$ with positive inverse X_q^{-1} . Indeed, X_q^{-1} satisfies

$$(2.2) \quad X X_q^{-1} = X_q^{-1} X = q$$

and if we denote $(X_q^{-1})^r = X_q^{-r}$ for $r > 0$, then we have

$$X^r X_q^{-r} = X_q^{-r} X^r = q.$$

The second one is the equality condition for the Hölder's inequality for semifinite von Neumann algebras.

Proposition 2.1. *Let $1 \leq p \leq \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$ and \mathcal{M} be a semifinite von Neumann algebra with a n.s.f. trace τ . If $a \in L_p(\mathcal{M})$ and $b \in L_{p'}(\mathcal{M})$ satisfy*

$$\|ab\|_{L_1(\mathcal{M})} = \|a\|_{L_p(\mathcal{M})} \|b\|_{L_{p'}(\mathcal{M})},$$

then we have

- (i) $(1 < p < \infty)$ $(bb^*)^{\frac{p'}{2}} = C \cdot (a^*a)^{\frac{p}{2}}$ for some constant C .
- (ii) $(p = 1)$ $qbb^*q = \|b\|^2 q$, where $q = \text{supp}(a)$.
- (iii) $(p = \infty)$ $Qaa^*Q = \|a\|^2 Q$, where $Q = \text{supp}(b^*)$.

Proof. The case $1 < p < \infty$ can be obtained from the Young's inequality for semifinite von Neumann algebras by Farenick and Manjegani in [4]. Actually, their results was only dealing with positive elements $a, b \in \mathcal{M}$, but we can apply the same proof for the general case. Note that every property of generalized singular values of the elements in $L_p(\mathcal{M})$ which we need in the proof can be found in [3].

Now we consider the case $p = 1$. Let $u|a|$ be the polar decomposition of a . Then we have

$$\begin{aligned}\tau(|ab|) &= \tau\left(\left|u|a|^{\frac{1}{2}} \cdot |a|^{\frac{1}{2}}b\right|\right) \leq \tau(|a|^{\frac{1}{2}}u^*u|a|^{\frac{1}{2}})^{\frac{1}{2}} \cdot \tau(|a|^{\frac{1}{2}}bb^*|a|^{\frac{1}{2}})^{\frac{1}{2}} \\ &\leq \tau(|a|) \cdot \|b\|.\end{aligned}$$

Since we are assuming that $\tau(|ab|) = \tau(|a|) \cdot \|b\|$ we can apply the case $p = 2$ to get

$$|a|^{\frac{1}{2}}bb^*|a|^{\frac{1}{2}} = C \cdot |a|$$

for some constant $C > 0$. As in (2.2) there is a positive operator $|a|_q^{-1}$ such that $|a||a|_q^{-1} = |a|_q^{-1}|a| = q$. By multiplying $|a|_q^{-\frac{1}{2}}$ on both sides we get

$$qbb^*q = C \cdot q.$$

Clearly, $C = \|qb\|^2 \leq \|b\|^2$. Since $aq = a$ and

$$\|a\|_{L_1(\mathcal{M})} \|b\| = \|ab\|_{L_1(\mathcal{M})} = \|a \cdot qb\|_{L_1(\mathcal{M})} \leq \|a\|_{L_1(\mathcal{M})} \|qb\|$$

we get the reverse inequality, which lead us to the conclusion.

The proof for the case $p = \infty$ is similar. □

The third one is Haagerup's reduction, which is very useful when we want to extend some results holding for finite von Neumann algebras to the general von Neumann algebra case. Let \mathcal{M} be a von Neumann algebra and $1 \leq p < \infty$. Then by Haagerup's reduction theorem (see [6, 25] for the details) there are increasing net $(\mathcal{M}_\alpha)_{\alpha \in \mathcal{I}}$ of finite von Neumann subalgebras of \mathcal{M} for a directed index set \mathcal{I} and complete contractions

$$u_{\alpha,p} : L_p(\mathcal{M}_\alpha) \rightarrow L_p(\mathcal{M}) \text{ and } w_{\alpha,p} : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}_\alpha)$$

such that

$$\lim_{\alpha} u_{\alpha,p} w_{\alpha,p}(x) = x \text{ for all } x \in L_p(\mathcal{M}).$$

Now we choose an ultra-filter \mathcal{U} associated to \mathcal{I} . Then

$$U = (u_{\alpha,p})_{\alpha} : \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}_\alpha) \rightarrow \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M})$$

and

$$W = (w_{\alpha,p})_{\alpha} : \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}) \rightarrow \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}_\alpha)$$

are complete contractions satisfying

$$(2.3) \quad UW|_{L_p(\mathcal{M})} = I_{L_p(\mathcal{M})}.$$

Note that $L_p(\mathcal{M})$ canonically embedded in $\prod_{\mathcal{U}, \alpha} L_p(\mathcal{M})$.

The fourth one is about the operator space structure of $L_p(\mathcal{M})^{op}$, the opposite of $L_p(\mathcal{M})$. We start with a basic observation about E^{op} , the opposite of an operator space E . Recall that E^{op} is the same space as E and we will denote the elements in E^{op} by x^t , so that

$$E \rightarrow E^{op}, x \mapsto x^t$$

is the formal identity. However, E^{op} is equipped with the following operator space structures.

$$\|(x_{ij}^t)\|_{M_n(E^{op})} := \|(x_{ji})\|_{M_n(E)}$$

for any $n \in \mathbb{N}$ and $(x_{ij}) \in M_n(E)$. By Theorem 1.5 of [17] we can easily see that

$$(2.4) \quad \|(x_{ij}^t)\|_{S_p^n(E^{op})} = \|(x_{ji})\|_{S_p^n(E)}$$

for any $n \in \mathbb{N}$ and $(x_{ij}) \in S_p^n(E)$. When \mathcal{A} is a C^* -algebra, \mathcal{A}^{op} can be understood as a C^* -algebra with the reversed multiplication, which implies that

$$x^t y^t = (yx)^t \text{ for any } x, y \in \mathcal{A}.$$

Suppose \mathcal{M} is σ -finite, and ϕ a normal faithful (shortly n.f.) finite weight on \mathcal{M} with density D_ϕ . Let ψ is the n.f. finite weight on \mathcal{M}^{op} defined by $\psi(x^t) = \phi(x)$ for any $x \in \mathcal{M}$, and let D_ψ be the density of ψ . Then, clearly $\mathcal{M}^{op} \rightarrow \mathcal{M}$, $x^t \rightarrow x$ is an isometry, and this isometry can be transferred to their preduals. Since we have

$$\text{tr}_{\mathcal{M}^{op}}(D_\psi x^t y^t) = \psi(x^t y^t) = \psi((yx)^t) = \phi(yx) = \text{tr}_{\mathcal{M}}(yx D_\phi)$$

for any $x, y \in \mathcal{M}$, $L_1(\mathcal{M}^{op})$ and $L_1(\mathcal{M})$ are isometric under the mapping

$$D_\psi x^t \mapsto x D_\phi.$$

Consequently, $L_p(\mathcal{M}^{op})$ are $L_p(\mathcal{M})$ isometric under the mapping

$$D_\psi^{\frac{1}{p}} x^t \mapsto x D_\phi^{\frac{1}{p}}$$

by complex interpolation for $1 < p < \infty$ ([1, 11]).

Furthermore, by applying the above isometry arising from $M_n \otimes \mathcal{M}$ with n.f. finite weight $\text{Tr} \otimes \phi$, where Tr is the usual trace, we have for any $(x_{ij})_{i,j=1}^n \in M_n(\mathcal{M})$ that

$$\begin{aligned} \left\| \sum_{i,j=1}^n e_{ij} \otimes (x_{ij} D_\phi^{\frac{1}{p}})^t \right\|_{S_p^n(L_p(\mathcal{M})^{op})} &= \left\| \sum_{i,j=1}^n e_{ij} \otimes x_{ji} D_\phi^{\frac{1}{p}} \right\|_{L_p(M_n \otimes \mathcal{M})} \\ &= \left\| \sum_{i,j=1}^n e_{ij}^t \otimes D_\psi^{\frac{1}{p}} x_{ji}^t \right\|_{L_p(M_n^{op} \otimes \mathcal{M}^{op})} \\ &= \left\| \sum_{i,j=1}^n e_{ij} \otimes D_\psi^{\frac{1}{p}} x_{ij}^t \right\|_{S_p^n(L_p(\mathcal{M}^{op}))}. \end{aligned}$$

The last equality is because of the fact $M_n^{op} \rightarrow M_n$, $e_{ij}^t \mapsto e_{ji}$ is a $*$ -isomorphism which can be extended to a complete isometry $L_p(M_n^{op}) \rightarrow S_p^n$, $e_{ij}^t \mapsto e_{ji}$ as above. Now, by (2.4) we get the following.

Proposition 2.2. *Let \mathcal{M} , ϕ , and ψ as before. Then we have the following complete isometry*

$$L_p(\mathcal{M}^{op}) \cong L_p(\mathcal{M})^{op}, \quad D_\psi^{\frac{1}{p}} x^t \mapsto (x D_\phi^{\frac{1}{p}})^t.$$

We end this section with the following automatic complete boundedness result.

Proposition 2.3. *Let \mathcal{M} be a von Neumann algebra, $2 \leq p < \infty$ and*

$$T : L_p(\mathcal{M}) \rightarrow RC_{p'}.$$

Then we have

$$\|T\|_{cb} \leq c_p \|T\|$$

for some constant $c_p > 0$ depending only on p .

Proof. We can still apply the proof of Proposition 4.2.6 of [5]. The only point we need to check is that

$$[(R_n \cap C_n) \otimes_{\min} \mathcal{M}, \ell_2^n \otimes_2 L_2(\mathcal{M})]_{\frac{1}{p}} \cong \mathcal{G}_p^n(L_p(\mathcal{M}))$$

isomorphically with constant c_p depending only on p , where \otimes_{\min} and \otimes_2 are injective tensor product of operator spaces and Hilbert space tensor product, respectively. Here, $\mathcal{G}_p^n(L_p(\mathcal{M}))$ implies

$$\text{span}\left\{\sum_{i=1}^n g_i \otimes x_i : x_i \in L_p(\mathcal{M})\right\} \subseteq L_p(\Omega; L_p(\mathcal{M})),$$

where $(g_i)_{i \geq 1}$ be a family of independent gaussian variables on a probability space (Ω, P) .

Let $(W_i)_{i \geq 1}$ be a family of free semi-circular variables on \mathbb{M} , the von Neumann algebra they generate and

$$\mathcal{W}_p^n(L_p(\mathcal{M})) = \text{span}\left\{\sum_{i=1}^n W_i \otimes x_i : x_i \in L_p(\mathcal{M})\right\} \subseteq L_p(\mathbb{M} \bar{\otimes} \mathcal{M}).$$

It is well known that ([26])

$$(R_n \cap C_n) \otimes_{\min} \mathcal{M} \cong \mathcal{W}_{\infty}^n(\mathcal{M}) \text{ and } \ell_2^n \otimes_2 L_2(\mathcal{M}) \cong \mathcal{W}_2^n(L_2(\mathcal{M}))$$

isomorphically with universal constants, and it is also well known that $\mathcal{W}_p^n(L_p(\mathcal{M}))$ is complemented in $L_p(\mathbb{M} \bar{\otimes} \mathcal{M})$ for all $2 \leq p \leq \infty$ with a universal constant ([26]). Thus, we get

$$[(R_n \cap C_n) \otimes_{\min} \mathcal{M}, \ell_2^n \otimes_2 L_2(\mathcal{M})]_{\frac{1}{p}} \cong \mathcal{W}_p^n(L_p(\mathcal{M}))$$

isomorphically with a universal constant. Since $\mathcal{G}_p^n(L_p(\mathcal{M})) \cong \mathcal{W}_p^n(L_p(\mathcal{M}))$ isomorphically with the constant depending only on p ([26, 8]) we are done. \square

3. CHANGE OF DENSITY

In this section we will observe a change of density phenomenon in the noncommutative context. We start with the following basis selection problem of independent interest. We basically follow the approach in Proposition III.B.7 of [24] with a suitable modification. However, we need to be more careful, since we are dealing with the case where the square function could be singular, which was easy to handle in the commutative case.

Proposition 3.1. *Let $1 \leq p < \infty$ and E be a n -dimensional subspace of $L_p(\mathcal{M})$, where \mathcal{M} is semifinite von Neumann algebra equipped with a n.s.f. trace τ . Then there is a basis $(x_i)_{i=1}^n$ of E with*

$$(3.1) \quad \tau(X^p) = n,$$

where $X = (\sum_{i=1}^n x_i^* x_i)^{\frac{1}{2}}$ satisfying the following. When $2 \leq p < \infty$ we have

$$(3.2) \quad \tau(X^{p-2} x_i^* x_j) = \delta_{ij} \text{ for any } 1 \leq i, j \leq n.$$

When $1 \leq p < 2$ we have

$$(3.3) \quad \tau(X_q^{p-2} x_i^* x_j) = \delta_{ij} \text{ for any } 1 \leq i, j \leq n.$$

where X_q^{-1} is the inverse of X as an operator affiliated to $q\mathcal{M}q$ and $q = \text{supp} X$ as in (2.2).

Proof. First, we find the solution of the following extremal problem:

given linearly independent $(\varphi_i)_{i=1}^n \subseteq E^*$, find $(x_i)_{i=1}^n \subseteq E$ giving

$$\max\{\det[\varphi_i(x_j^*)]_{i,j=1}^n : \left\| \left(\sum_{i=1}^n x_i^* x_i \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M})} \leq n^{\frac{1}{p}}\}.$$

Since E is finite dimensional the solution $(x_i)_{i=1}^n \subseteq E$ exists. Note that if $(x_i)_{i=1}^n \subseteq E$ gives the maximum for a sequence $(\varphi_i)_{i=1}^n \subseteq E^*$, then it gives the maximum for any linearly independent $(\psi_i)_{i=1}^n \subseteq E^*$. Thus, we can choose $(\psi_i)_{i=1}^n \subseteq E^*$ satisfying $\psi_i(x_j^*) = \delta_{ij}$ for any $1 \leq i, j \leq n$.

Since

$$\left\| \left(\sum_{i=1}^n x_i^* x_i \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{M})} = \left\| \sum_{i=1}^n x_i \otimes e_{i1} \right\|_{L_p(\mathcal{M}; C_p^n)}$$

and $L_p(\mathcal{M}; C_p^n)^* = L_{p'}(\mathcal{M}; C_{p'}^n)$, with the same argument as in the proof of Proposition III.B.7 of [24] we can find extensions $(y_i)_{i=1}^n \subseteq L_{p'}(\mathcal{M})$ of $(\psi_i)_{i=1}^n \subseteq E^*$ satisfying

$$\left\| \sum_{i=1}^n y_i \otimes e_{i1} \right\|_{L_{p'}(\mathcal{M}; C_{p'}^n)} = \left\| \left(\sum_{i=1}^n y_i^* y_i \right)^{\frac{1}{2}} \right\|_{L_{p'}(\mathcal{M})} = n^{\frac{1}{p'}}.$$

Then we have

$$\begin{aligned} n &= \langle (x_i^*), (y_i) \rangle = \sum_{i=1}^n \tau(x_i^* y_i) = \tau \left(\sum_{i=1}^n x_i y_i^* \right) \\ &= \tau \otimes \text{Tr} \left[\left(\sum_i x_i \otimes e_{i1} \right) \left(\sum_j y_j^* \otimes e_{1j} \right) \right] = \tau \otimes \text{Tr}(TS^*) \\ &\leq \left\| \sum_{i=1}^n x_i \otimes e_{i1} \right\|_{L_p(\mathcal{M} \otimes M_n)} \left\| \sum_{j=1}^n y_j^* \otimes e_{1j} \right\|_{L_{p'}(\mathcal{M} \otimes M_n)} \\ &= \left\| \left(\sum_{i=1}^n x_i^* x_i \right)^{\frac{1}{2}} \right\|_p \left\| \left(\sum_{i=1}^n y_i^* y_i \right)^{\frac{1}{2}} \right\|_{p'} = n. \end{aligned}$$

If $2 \leq p < \infty$, then by Proposition 2.1 we get

$$|T|^p = X^p \otimes e_{11} = C \cdot |S|^{p'} = C \cdot Y^{p'} \otimes e_{11}$$

for some constant $C > 0$, where $Y = (\sum_{i=1}^n y_i^* y_i)^{\frac{1}{2}}$. Since

$$\tau(|T|^p) = \tau(X^p) \cdot \text{Tr}(e_{11}) = n \text{ and } \tau(|S|^{p'}) = \tau(Y^{p'}) \cdot \text{Tr}(e_{11}) = n$$

we get

$$X^p = Y^{p'} \text{ or equivalently } Y = X^{p-1}.$$

Now we set $q = \text{supp} X$ and consider X_q^{-1} as before. Since $X^2 = \sum_{i=1}^n |x_i|^2$ we have

$$\text{supp} X = \text{supp} X^2 = \bigvee_{i=1}^n \text{supp} |x_i|^2 = \bigvee_{i=1}^n \text{supp} |x_i|,$$

so that we have

$$(3.4) \quad q |x_i| = |x_i| q = |x_i|$$

for any $1 \leq i \leq n$. Similarly, we have $q |y_i| = |y_i| q = |y_i|$ for any $1 \leq i \leq n$. Thus, we have

$$1 = \tau(x_i^* y_i) = \tau(x_1^* y_i q) = \tau(X^{\frac{p}{2}-1} x_1^* y_i X_q^{1-\frac{p}{2}})$$

for any $1 \leq i \leq n$, so that

$$\begin{aligned} (3.5) \quad n &= \tau \left(\frac{1}{2} \left[\sum_{i=1}^n X^{\frac{p}{2}-1} x_i^* y_i X_q^{1-\frac{p}{2}} + \sum_{i=1}^n X_q^{1-\frac{p}{2}} y_i^* x_i X^{\frac{p}{2}-1} \right] \right) \\ &\leq \tau \left(\frac{1}{2} \left[\sum_{i=1}^n X^{\frac{p}{2}-1} x_i^* x_i X^{\frac{p}{2}-1} + \sum_{i=1}^n X_q^{1-\frac{p}{2}} y_i^* y_i X_q^{1-\frac{p}{2}} \right] \right) \\ &= \frac{1}{2} \tau(X^p) + \frac{1}{2} \tau(X^{2p-2} X_q^{2-p}) = \frac{1}{2} \tau(X^p) + \frac{1}{2} \tau(X^p q) \\ &= \tau(X^p) = n. \end{aligned}$$

Thus, equality should hold in the above inequality, which means that

$$x_i X^{\frac{p}{2}-1} = y_i X_q^{1-\frac{p}{2}}, \text{ or equivalently, } y_i = y_i q = x_i X^{p-2} \text{ for any } 1 \leq i \leq n.$$

Consequently, we have

$$\tau(X^{p-2} x_j^* x_i) = \tau(y_j^* x_i) = \delta_{ij} \text{ for all } 1 \leq i, j \leq n.$$

When $1 < p < 2$, we just replace $x_i X^{\frac{p}{2}-1}$ and $y_i X_q^{1-\frac{p}{2}}$ by $x_i X_q^{\frac{p}{2}-1}$ and $y_i X^{1-\frac{p}{2}}$, respectively, in (3.5), then we get $y_i = x_i X_q^{p-2}$ for any $1 \leq i \leq n$ by the same argument, which leads us to the desired conclusion.

If $p = 1$, then again by Proposition 2.1 we get

$$Q(Y^2 \otimes e_{11})Q = QS^*SQ = \|S\|^2 Q = Q \text{ for } Q = \text{supp}T.$$

Since $X^{\frac{1}{2}} \otimes e_{11} = |T|^{\frac{1}{2}}$ we have $Q(X^{\frac{1}{2}} \otimes e_{11})Q = X^{\frac{1}{2}} \otimes e_{11}$, so that

$$\begin{aligned} \tau(X^{\frac{1}{2}} Y^2 X^{\frac{1}{2}}) &= \tau \otimes \text{Tr}[(X^{\frac{1}{2}} \otimes e_{11}) \cdot (Y^2 \otimes e_{11}) \cdot (X^{\frac{1}{2}} \otimes e_{11})] \\ &= \tau \otimes \text{Tr}[Q(X^{\frac{1}{2}} \otimes e_{11})Q \cdot Q(Y^2 \otimes e_{11})Q \cdot Q(X^{\frac{1}{2}} \otimes e_{11})Q] \\ &= \tau \otimes \text{Tr}[Q(X^{\frac{1}{2}} \otimes e_{11})Q \cdot Q \cdot Q(X^{\frac{1}{2}} \otimes e_{11})Q] \\ &= \tau(X). \end{aligned}$$

Thus,

$$\begin{aligned} n &= \tau\left(\frac{1}{2}\left[\sum_{i=1}^n X_q^{-\frac{1}{2}} x_i^* y_i X^{\frac{1}{2}} + \sum_{i=1}^n X^{\frac{1}{2}} y_i^* x_i X_q^{-\frac{1}{2}}\right]\right) \\ &\leq \tau\left(\frac{1}{2}\left[\sum_{i=1}^n X_q^{-\frac{1}{2}} x_i^* x_i X_q^{-\frac{1}{2}} + \sum_{i=1}^n X^{\frac{1}{2}} y_i^* y_i X^{\frac{1}{2}}\right]\right) \\ &= \frac{1}{2}\tau(qX) + \frac{1}{2}\tau(X^{\frac{1}{2}} Y^2 X^{\frac{1}{2}}) = \tau(X) = n, \end{aligned}$$

and the rest of the argument is the same. \square

With the previous proposition we can show the following factorization results.

Proposition 3.2. *Let \mathcal{M} be as before and E be a n -dimensional subspace of $L_p(\mathcal{M})$ for $1 < p < \infty$.*

(i) *When $2 \leq p < \infty$ we have*

$$L_p(\mathcal{M}) \xrightarrow{A} C_p^n \xrightarrow{B} E \text{ such that } \begin{cases} BA|_E = I_E, \\ \|B\|_{cb} \leq 2^{\frac{1}{2}-\frac{1}{p}} \text{ and} \\ \|A\| \leq n^{\frac{1}{2}-\frac{1}{p}}. \end{cases}$$

(ii) *When $1 < p < 2$ we have*

$$L_p(\mathcal{M}) \xrightarrow{A} C_p^n \xrightarrow{B} E \text{ such that } \begin{cases} BA|_E = I_E, \\ \|B\| \leq n^{\frac{1}{p}-\frac{1}{2}} \text{ and} \\ \|A\|_{cb} \leq 2^{\frac{1}{p}-\frac{1}{2}}. \end{cases}$$

Proof. (i) By Proposition 3.1 we can find a basis $(x_i)_{i=1}^n$ of E satisfying (3.1) and (3.2). Let φ be a weight on \mathcal{M} defined by

$$\varphi(\cdot) = \tau(\cdot X^{p-2}).$$

Let $\mathcal{N}_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}$, $N_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) = 0\}$, and $L_2(\varphi)$ be the Haagerup L_2 -space defined by the completion of $\mathcal{N}_\varphi/N_\varphi$ with respect to the inner product

$$\langle x + N_\varphi, y + N_\varphi \rangle_{L_2(\varphi)} = \varphi(y^*x).$$

Then, $(x_i + N_\varphi)_{i=1}^n$ is a orthonormal sequence in $L_2(\varphi)$.

Now we consider the following factorization of a projection onto E .

$$L_p(\mathcal{M}) \xrightarrow{w_1} L_2(\varphi) \xrightarrow{Q_\varphi} E_\varphi \xrightarrow{w_2} E,$$

where $w_1(x) = x + N_\varphi$ for $x \in \mathcal{N}_\varphi$, $E_\varphi = w_1(E)$, Q_φ is the orthogonal projection onto E_φ , and $w_2(x_i + N_\varphi) = x_i$ for any $1 \leq i \leq n$. Then, we have

$$\|w_1\| \leq n^{\frac{1}{2} - \frac{1}{p}} \text{ and } \|w_2 : E_\varphi^{C_p} \rightarrow E\|_{cb} \leq 2^{\frac{1}{2} - \frac{1}{p}},$$

where $E_\varphi^{C_p}$ is E_φ equipped with C_p structure. Note that $E_\varphi^{C_p} \cong C_p^n$ completely isometrically.

Indeed, for any $x \in \mathcal{N}_\varphi$ we have

$$\|x + N_\varphi\|_{L_2(\varphi)} = \tau(X^{p-2}|x|^2)^{\frac{1}{2}} \leq \tau(X^{(p-2)r})^{\frac{1}{2r}} \tau(|x|^p)^{\frac{1}{p}} = n^{\frac{1}{2} - \frac{1}{p}} \cdot \|x\|_{L_p(\mathcal{M})}$$

by Hölder's inequality and (3.1), where $\frac{1}{r} + \frac{2}{p} = 1$. Furthermore, for any $m \in \mathbb{N}$ and $(\alpha_i)_{i=1}^n \in M_m$ we consider $T = \sum_{i=1}^n \alpha_i \otimes x_i$ and $A = \left(\sum_{i=1}^n \alpha_i^* \alpha_i\right)^{\frac{p-2}{2}}$, then we have

$$\begin{aligned} \text{Tr} \otimes \tau \left(\left| \sum_{i=1}^n \alpha_i \otimes x_i \right|^p \right) &= \text{Tr} \otimes \tau(|T|^{p-2} |T|^2) \\ &\leq 2^{\frac{p}{2}-1} \cdot \text{Tr} \otimes \tau([A \otimes X^{p-2}] |T|^2) \\ &= 2^{\frac{p}{2}-1} \cdot \text{Tr} \otimes \tau \left(\sum_{i,j=1}^n A \alpha_i^* \alpha_j \otimes X^{p-2} x_i^* x_j \right) \\ &= 2^{\frac{p}{2}-1} \cdot \sum_{i,j=1}^n \text{Tr}(A \alpha_i^* \alpha_j) \tau(X^{p-2} x_i^* x_j) \\ &= 2^{\frac{p}{2}-1} \cdot \text{Tr} \left(\left[\sum_{i=1}^n \alpha_i^* \alpha_i \right]^{\frac{p}{2}} \right) \end{aligned}$$

by (3.2) and (ii) of Lemma 3.2 of [12], since we have

$$(3.6) \quad \left| \sum_{i=1}^n \alpha_i \otimes x_i \right|^2 \leq 2 \left(\sum_{i=1}^n \alpha_i^* \alpha_i \right) \otimes \left(\sum_{i=1}^n x_i^* x_i \right).$$

Now, if we take $A = Q_\varphi w_1$ and $B = w_2$ we are done.

(ii) We take a basis $(x_i)_{i=1}^n$ of E as in Proposition 3.1 satisfying (3.1) and (3.3), and set φ be a weight on \mathcal{M} defined by

$$\varphi(\cdot) = \tau(\cdot X_q^{p-2}).$$

We consider \mathcal{N}_φ , N_φ , $L_2(\varphi)$, and the following factorization of a projection onto E as before.

$$L_p(\mathcal{M}) \xrightarrow{w_1} L_2(\varphi) \xrightarrow{Q_\varphi} E_\varphi \xrightarrow{w_2} E \xhookrightarrow{j} L_p(\mathcal{M}),$$

where j is the inclusion. Then, we claim that

$$\|w_1 : L_p(\mathcal{M}) \rightarrow E_\varphi^{C_p}\|_{cb} \leq 2^{\frac{1}{p} - \frac{1}{2}} \text{ and } \|w_2\| \leq n^{\frac{1}{p} - \frac{1}{2}}.$$

We consider the adjoint maps to check the above claims. We have

$$(jw_2)^*(\xi) = \xi X^{2-p} + N_\varphi \text{ for } \xi \in \mathcal{N}_\varphi.$$

Indeed, by (3.4) we have $qx_i^* = x_i^*$ for any $1 \leq i \leq n$, so that

$$\langle (jw_2)^*(\xi), x_i + N_\varphi \rangle = \langle \xi, jw_2(x_i + N_\varphi) \rangle = \tau(x_i^* \xi) = \varphi(x_i^* \xi X^{2-p})$$

for any $1 \leq i \leq n$. Then, $\|(jw_2)^*\| \leq n^{\frac{1}{p}-\frac{1}{2}}$ since

$$\begin{aligned} \|\xi X^{2-p} + N_\varphi\|_{L_2(\varphi)} &= \tau(X^{2-p} \xi^* \xi X^{2-p} X_q^{p-2})^{\frac{1}{2}} = \tau(q X^{2-p} |\xi|^2)^{\frac{1}{2}} = \tau(X^{2-p} |\xi|^2)^{\frac{1}{2}} \\ &\leq \tau(X^{(2-p)r})^{\frac{1}{2r}} \tau(|\xi|^{p'})^{\frac{1}{p'}} = \tau(X^p)^{\frac{1}{2r}} \|\xi\|_{L_p(\mathcal{M})} \\ &= n^{\frac{1}{p}-\frac{1}{2}} \cdot \|\xi\|_{L_p(\mathcal{M})} \end{aligned}$$

by (3.1), where $\frac{1}{r} + \frac{2}{p'} = 1$.

Furthermore, we have $w_1^*(\eta + N_\varphi) = \eta X_q^{p-2}$ for any $\eta \in \mathcal{N}_\varphi$ since

$$\langle w_1^*(\eta + N_\varphi), \xi \rangle = \langle \eta + N_\varphi, w_1 \xi \rangle_{L_2(\varphi)} = \tau(\xi^* \eta X_q^{p-2})$$

for any $\xi \in \mathcal{N}_\varphi$. Then $\|(Q_\varphi w_1)^* : E_\varphi^{C_{p'}} \rightarrow L_{p'}(\mathcal{M})\|_{cb} \leq 2^{\frac{1}{p}-\frac{1}{2}}$. Indeed, for any $m \in \mathbb{N}$ and $(\alpha_i)_{i=1}^n \in M_m$ we set $T = \sum_{i=1}^n \alpha_i \otimes x_i X_q^{p-2}$ and $A = \left(\sum_{i=1}^n \alpha_i^* \alpha_i \right)^{\frac{p'-2}{2}}$, then we have by (3.6), (ii) of Lemma 3.2 of [12], (3.4), and (3.3) that

$$\begin{aligned} \text{Tr} \otimes \tau \left(\left| \sum_{i=1}^n \alpha_i \otimes x_i X_q^{p-2} \right|^{p'} \right) &= \text{Tr} \otimes \tau (|T|^{p'-2} |T|^2) \\ &\leq 2^{\frac{p'}{2}-1} \cdot \text{Tr} \otimes \tau \left([A \otimes (X_q^{p-2} X^2 X_q^{p-2})^{\frac{p'-2}{2}}] |T|^2 \right) \\ &= 2^{\frac{p'}{2}-1} \cdot \text{Tr} \otimes \tau \left([A \otimes X^{(p-1)(p'-2)}] |T|^2 \right) \\ &= 2^{\frac{p'}{2}-1} \cdot \text{Tr} \otimes \tau \left(\sum_{i,j=1}^n A \alpha_i^* \alpha_j \otimes q x_i^* x_j X_q^{p-2} \right) \\ &= 2^{\frac{p'}{2}-1} \cdot \text{Tr} \otimes \tau \left(\sum_{i,j=1}^n A \alpha_i^* \alpha_j \otimes x_i^* x_j X_q^{p-2} \right) \\ &= 2^{\frac{p'}{2}-1} \cdot \text{Tr} \left(\left[\sum_{i=1}^n \alpha_i^* \alpha_i \right]^{\frac{p'}{2}} \right). \end{aligned}$$

Now, if we take $A = Q_\varphi w_1$ and $B = w_2$ we get the desired result. \square

Proposition 3.3. *Proposition 3.2 is true for a general von Neumann algebra \mathcal{M} .*

Proof. We only check the case $2 \leq p < \infty$, since the other case can be shown similarly. Let U and W are the maps in (2.3). Now we fix a basis $(x_i)_{i=1}^n$ of E and choose a representative $(x_{\alpha,i})_\alpha$ of $Wx_i \in \prod_{\mathcal{U},\alpha} L_p(\mathcal{M}_\alpha)$ with $x_{\alpha,i} \in L_p(\mathcal{M}_\alpha)$ for $1 \leq i \leq n$ (Note that $E \hookrightarrow \prod_{\mathcal{U},\alpha} L_p(\mathcal{M})$ canonically). If we set

$$E_\alpha = \text{span}\{x_{\alpha,i}\}_{i=1}^n \subseteq L_p(\mathcal{M}_\alpha),$$

then $\dim E_\alpha \leq n$ and by Proposition (3.2) we get

$$L_p(\mathcal{M}) \xrightarrow{A_\alpha} C_p^n \xrightarrow{B_\alpha} E_\alpha \text{ such that } \begin{cases} B_\alpha A_\alpha|_{E_\alpha} = I_{E_\alpha}, \\ \|B_\alpha\|_{cb} \leq 2^{\frac{1}{2}-\frac{1}{p}} \text{ and} \\ \|A_\alpha\| \leq n^{\frac{1}{2}-\frac{1}{p}}. \end{cases}$$

By setting $\tilde{A} = (A_\alpha)_\alpha$ and $\tilde{B} = (B_\alpha)_\alpha$ we get the following factorization of a projection onto E .

$$L_p(\mathcal{M}) \xrightarrow{W} \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}_\alpha) \xrightarrow{\tilde{A}} \prod_{\mathcal{U}, \alpha} C_p^n \xrightarrow{\tilde{B}} \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}_\alpha) \xrightarrow{U} \prod_{\mathcal{U}, \alpha} L_p(\mathcal{M}).$$

Note that $\prod_{\mathcal{U}, \alpha} C_p^n \cong C_p^n$ completely isometrically. Moreover, we have

$$U\tilde{B}\tilde{A}W|_E = I_E, \quad \|U\tilde{B}\|_{cb} \leq 2^{\frac{1}{2}-\frac{1}{p}} \quad \text{and} \quad \|\tilde{A}W\| \leq n^{\frac{1}{2}-\frac{1}{p}}.$$

□

Proposition 3.4. *Let \mathcal{M} be a von Neumann algebra and E be a n -dimensional quotient space of $L_p(\mathcal{M})$ with the quotient map $Q : L_p(\mathcal{M}) \rightarrow E$.*

(i) *When $2 \leq p < \infty$ we have*

$$E \xrightarrow{A} C_p^n \xrightarrow{B} L_p(\mathcal{M}) \xrightarrow{Q} E \text{ such that } \begin{cases} QBA|_E = I_E, \\ \|B\|_{cb} \leq 2^{\frac{1}{2}-\frac{1}{p}} \text{ and} \\ \|A\| \leq n^{\frac{1}{2}-\frac{1}{p}}. \end{cases}$$

(ii) *When $1 < p < 2$ we have*

$$E \xrightarrow{A} C_p^n \xrightarrow{B} L_p(\mathcal{M}) \xrightarrow{Q} E \text{ such that } \begin{cases} QBA|_E = I_E, \\ \|B\| \leq n^{\frac{1}{p}-\frac{1}{2}} \text{ and} \\ \|A\|_{cb} \leq 2^{\frac{1}{p}-\frac{1}{2}}. \end{cases}$$

Proof. We only check the case $1 < p < 2$ since the other case can be shown similarly. Since E^* is a n -dimensional subspace of $L_{p'}(\mathcal{M})$ by the embedding $j : E^* \hookrightarrow L_{p'}(\mathcal{M})$ we have the following factorization of the identity map on E^* by Proposition 3.3.

$$E^* \xrightarrow{j} L_{p'}(\mathcal{M}) \xrightarrow{C} C_{p'}^n \xrightarrow{D} E^*$$

with $DCj = I_{E^*}$, $\|D\| \leq n^{\frac{1}{2}-\frac{1}{p}}$ and $\|C\|_{cb} \leq 2^{\frac{1}{2}-\frac{1}{p}}$. By taking adjoint and setting $A = D^*$ and $B = C^*$ we are done. □

4. MAIN RESULTS

Finally, we present the main theorem of this paper. The proof can be established by the opposite trick (Corollary 7.9 of [18] or Proposition 4.2.6 of [5]).

Theorem 4.1. *Let E be n -dimensional subspace of $L_p(\mathcal{M})$ for a von Neumann algebra \mathcal{M} . Then we have*

$$d_{cb}(E, RC_{p'}^n) \leq c_p \cdot n^{|\frac{1}{2}-\frac{1}{p}|}$$

for some constant c_p depending only on p .

Proof. First of all, we can assume that \mathcal{M} is σ -finite by the usual density argument (Theorem 3.2 of [7]), and we consider the case $2 \leq p < \infty$. By the discussion in section 2, we can assume that $E^{op} \subseteq L_p(\mathcal{M})^{op}$. Since $E \cap E^{op} \cong E \cap_p E^{op}$ completely 2-isomorphic and

$$E \cap_p E^{op} \subseteq L_p(\mathcal{M}) \oplus_p L_p(\mathcal{M}^{op}) = L_p(\mathcal{M} \oplus \mathcal{M}^{op})$$

we have by Proposition 3.3 the following maps.

$$L_p(\mathcal{M} \oplus \mathcal{M}^{op}) \xrightarrow{A} C_p^n \xrightarrow{B} E \cap E^{op} \text{ such that } \begin{cases} BA|_{E \cap E^{op}} = I_{E \cap E^{op}}, \\ \|B\|_{cb} \leq 2 \cdot 2^{\frac{1}{2} - \frac{1}{p}} \text{ and} \\ \|A\| \leq n^{\frac{1}{2} - \frac{1}{p}}. \end{cases}$$

Clearly, we have $\|B : C_p^n \rightarrow E\|_{cb} \leq 2 \cdot 2^{\frac{1}{2} - \frac{1}{p}}$. Moreover, since R_p^n and $(C_p^n)^{op}$ are completely isometric we have

$$\|B : R_p^n \rightarrow E\|_{cb} = \|B : C_p^n \rightarrow E^{op}\|_{cb} \leq 2 \cdot 2^{\frac{1}{2} - \frac{1}{p}},$$

so consequently we get

$$(4.1) \quad \|B : R_p^n + C_p^n \rightarrow E\|_{cb} \leq 2 \cdot 2^{\frac{1}{2} - \frac{1}{p}}.$$

By applying Proposition 2.3 to A we get

$$\|A : L_p(\mathcal{M} \oplus \mathcal{M}^{op}) \rightarrow R_p^n + C_p^n\|_{cb} \leq c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}.$$

Thus, we have

$$\begin{aligned} \|T := (A|_{E \cap E^{op}})^* : R_p^n \cap C_p^n \rightarrow E^* \cap (E^*)^{op}\|_{cb} \\ = \|A|_{E \cap E^{op}} : E \cap E^{op} \rightarrow R_p^n + C_p^n\|_{cb} \leq c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}. \end{aligned}$$

Now let $(\delta_i)_{i=1}^n$ be the canonical basis for $R_p^n \cap C_p^n$. Then we have the decomposition $T\delta_i = a_i + b_i$ such that

$$\left\| \sum_{i=1}^n \delta_i \otimes a_i \right\|_{(R_p^n + C_p^n) \otimes_{\min} E^*} \leq c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}$$

and

$$\left\| \sum_{i=1}^n \delta_i \otimes b_i \right\|_{(R_p^n + C_p^n) \otimes_{\min} E^*} = \left\| \sum_{i=1}^n \delta_i \otimes b_i \right\|_{(R_p^n + C_p^n) \otimes_{\min} (E^*)^{op}} \leq c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}.$$

Thus, we have

$$\begin{aligned} (4.2) \quad \|T^* : E \rightarrow R_p^n + C_p^n\|_{cb} &= \|T : R_p^n \cap C_p^n \rightarrow E^*\| \\ &= \left\| \sum_{i=1}^n \delta_i \otimes (a_i + b_i) \right\|_{(R_p^n + C_p^n) \otimes_{\min} E^*} \\ &\leq 2 \cdot c_p \cdot n^{\frac{1}{2} - \frac{1}{p}}. \end{aligned}$$

Combining (4.1) and (4.2) we get the desired result.

Now we consider the case $1 < p < 2$. Since $E +_p E^{op}$ is a n -dimensional quotient space of $L_p(\mathcal{M} \oplus \mathcal{M}^{op})$ we can apply Proposition 3.4, so that we get the following factorization of $I_{E + E^{op}}$.

$$E + E^{op} \xrightarrow{A} C_p^n \xrightarrow{B} L_p(\mathcal{M} \oplus \mathcal{M}^{op}) \xrightarrow{Q} E + E^{op} \text{ such that } \begin{cases} QBA|_{E + E^{op}} = I_{E + E^{op}}, \\ \|B\| \leq n^{\frac{1}{p} - \frac{1}{2}} \text{ and} \\ \|A\|_{cb} \leq 2 \cdot 2^{\frac{1}{p} - \frac{1}{2}}. \end{cases}$$

As before, we can easily observe that

$$(4.3) \quad \|A : E \rightarrow R_p^n \cap C_p^n\|_{cb} \leq 2 \cdot 2^{\frac{1}{p} - \frac{1}{2}}.$$

By applying Proposition 2.3 to B^* we get

$$\|B : R_p^n \cap C_p^n \rightarrow L_p(\mathcal{M} \oplus \mathcal{M}^{op})\|_{cb} \leq c_p \cdot n^{\frac{1}{p}-\frac{1}{2}}$$

so that $\|S := QB : R_p^n \cap C_p^n \rightarrow E + E^{op}\|_{cb} \leq c_p \cdot n^{\frac{1}{p}-\frac{1}{2}}$. As before, we can show that

$$(4.4) \quad \|S : R_p^n \cap C_p^n \rightarrow E\|_{cb} \leq 2 \cdot c_p \cdot n^{\frac{1}{p}-\frac{1}{2}}.$$

Combining (4.3) and (4.4) we get the desired result. \square

We get an estimate of the relative cb-projection constant as a corollary using Proposition 2.3 of [14].

Corollary 4.2. *Let E be n -dimensional subspace of $L_p(\mathcal{M})$ for a von Neumann algebra \mathcal{M} . Then there is a projection $P : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$ onto E satisfying*

$$\gamma_{RC_{p'}}(P) \leq c_p \cdot n^{|\frac{1}{2}-\frac{1}{p}|}$$

for some constant c_p depending only on p . In particular, we have

$$\lambda_{cb}(E, L_p(\mathcal{M})) \leq c_p \cdot n^{|\frac{1}{2}-\frac{1}{p}|}.$$

In the above, $\gamma_{RC_{p'}}(\cdot)$ is the gamma-norm defined by

$$\gamma_{RC_{p'}}(T : E \rightarrow F) = \inf\{\|A\|_{cb} \|B\|_{cb}\},$$

where the infimum runs over all possible factorization $T : E \xrightarrow{A} RC_{p'}(I) \xrightarrow{B} F$ for some index set I , and $RC_{p'}(I)$ is $\ell_2(I)$ with the $RC_{p'}$ structure (p. 82 of [16]).

Remark 4.3. (1) When \mathcal{M} is commutative there is one more possible estimate. Let E be a n -dimensional subspace of $L_p(\mu)$ for some measure μ . Then, we have

$$d_{cb}(E, RC_p^n) \leq c_p \cdot n^{|\frac{1}{2}-\frac{1}{p}|}.$$

This is obtained directly from Proposition 3.3 and the following variation of Proposition 2.3, which can be shown by the same proof as Proposition 4.2.6 of [5].

$$\|T : L_p(\mu) \rightarrow RC_p\|_{cb} \leq c_p \|T\|$$

for some constant $c_p > 0$ depending only on p .

- (2) The main theorem in this paper deals with the cb-distance between a finite dimensional subspace of L_p and $RC_{p'}^n$. However, there is another operator space structure on ℓ_2^n in which we would naturally be interested, namely, OH_n , the n -dimensional operator Hilbert space by Pisier. Actually, we can show similar result for OH_n instead of $RC_{p'}^n$ using θ -Hilbertian approach. More precisely, we consider an operator space E and we fix $n \in \mathbb{N}$ and $C > 0$. Then the following are equivalent.

- (i) $\pi_{2,OH}^*(v) \leq C \cdot \pi_{2,OH}(v^*)$ for any $v : \ell_2^n \rightarrow E$.
- (ii) $\|A \otimes I_E : S_2(E) \rightarrow S_2(E)\| \leq C$ for any $A : S_2 \rightarrow S_2$ with rank $\leq n$.
- (iii) For any n -dimensional subspace F of E we have $P : E \rightarrow E$ with

$$P|_E = I_E \text{ and } \gamma_{oh}(P) \leq C.$$

- (iv) For any n -dimensional subspace F of E we have $d_{cb}(F, OH_n) \leq C$. Here, $\pi_{2,OH}(\cdot)$ is the $(2, OH)$ -summing norm defined by

$$\pi_{2,OH}(T : E \rightarrow \ell_2) = \sup \left\{ \frac{\|TS\|_{HS}}{\|S : OH \rightarrow E\|_{cb}} \right\}.$$

See [15] for the related topics. The equivalence between (i) and (ii) can be obtained by imitating the proof of Proposition 1 of [19] and Theorem 6.9

of [17]. (i) \Rightarrow (iii) by the proof of Proposition 2.3 of [14], and (iii) \Rightarrow (iv) \Rightarrow (i) is clear.

Note that we can easily apply complex interpolation to (ii), so that $L_p(\mathcal{M})$ satisfies the above conditions with $C = n^{|\frac{1}{2} - \frac{1}{p}|}$ since it is a complex interpolation of an operator space and an operator Hilbert space ($L_2(\mathcal{M})$) by the parameter $\theta = \min\{\frac{2}{p}, \frac{2}{p'}\}$.

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